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Moduli of stable sheaves on a smooth quadric and a Brill–Noether locus

Sukmoon Huh

Department of Mathematics, Sungkyunkwan University, Suwon, 440-746, Republic of Korea

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ABSTRACT

We prove that the moduli space of stable sheaves of rank 2 with the Chern classes $c_1 = \mathcal{O}_Q(1,1)$ and $c_2 = 2$ on a smooth quadric Q in $\mathbb{P}_3$ is isomorphic to $\mathbb{P}_3$. Using this identification, we give a new proof that a Brill–Noether locus, defined as the closure of the stable bundles with at least three linearly independent sections, on a non-hyperelliptic curve of genus 4, is isomorphic to the *Donagi–Izadi cubic threefold*.

1. Introduction

Throughout this paper, the ground field is always assumed to be $\mathbb{C}$, the field of complex numbers. Let $SU_C(2, K_C)$ be the moduli space of semi-stable vector bundles of rank 2 with canonical determinant over curves C. In [9], the geometry of the Brill–Noether loci $\mathcal{W}^r$ of $SU_C(2, K_C)$, the closure of the stable vector bundles $E \in SU_C(2, K_C)$ with $h^0(E) \geq r + 1$, was investigated. In [4], the explicit description of $\mathcal{W}^1$ when $g(C) = 3$ was rediscovered, using the rational map defined by the restriction from $\overline{M}(1, 2)$ to $\mathcal{W}^1$, where $\overline{M}(1, 2)$ is the moduli space of stable sheaves of rank 2 over $\mathbb{P}_2$ with Chern classes $c_1 = 1$ and $c_2 = 2$. In this article, we use the same method as in [4] to rediscover the geometry of $\mathcal{W}^2$ stated in [9], using the rational restriction map

$$\Phi : \overline{M}(2) \dashrightarrow \mathcal{W}^2,$$

sending E to $E|_C$, where $\overline{M}(2)$ is the moduli space of semi-stable sheaves of rank 2 with the Chern classes $c_1 = \mathcal{O}_Q(1, 1)$ and $c_2 = 2$ over a smooth quadric surface $Q \subset \mathbb{P}_3$, containing C, and $\mathcal{W}^2$ is a Brill–Noether locus of a non-hyperelliptic curve of genus 4. This rational map makes sense because C is canonically embedded into $\mathbb{P}_3$, and there exists a unique quadric surface containing it. We will deal with the case when Q is smooth, i.e., the pencils of the two trigonal line bundles of C do not coincide.

First, we give an explicit description of $\overline{M}(2)$. We construct a morphism $\Psi : \overline{M}(2) \to \mathbb{P}_3$, and prove that Ψ is an isomorphism. We then introduce the concept of the jumping conics of $E \in \overline{M}(2)$, the conics on Q over which the splitting type of E is not generic. We show that Ψ can be redefined by sending E to the set of jumping conics of E, which is a hyperplane in $\mathbb{P}_3^*$.

Last, we investigate the restriction map $\Phi : \overline{M}(2) \dashrightarrow \mathcal{W}^2$. We prove that this map is birational and given by the complete linear system $|I_C(3)|$. Here, $I_C(3)$ is the ideal sheaf of $C \subset \mathbb{P}_3 \simeq \overline{M}(2)$, twisted by $\mathcal{O}_{\mathbb{P}_3}(3)$. This will give us an easy proof that $\mathcal{W}^2$ is isomorphic to the *Donagi–Izadi cubic threefold* [9]. If we compose Φ with the projection $\mathcal{W}^2 \dashrightarrow \mathbb{P}_3^*$ at the unique point of $\mathcal{W}^3$, we have an isomorphism of $\overline{M}(2)$ with $\mathbb{P}_3^*$. We also describe this isomorphism in terms of the stable sheaves in $\overline{M}(2)$.

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E-mail addresses: sukmoonh@kias.re.kr, sukmoonh@skku.edu.

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2. Description of $\overline{M}(2)$

Let Q be a smooth quadric isomorphic to $\mathbb{P}(V_1)\times\mathbb{P}(V_2)$, embedded into $\mathbb{P}_3\simeq\mathbb{P}(V)$ by the Segre map, where $V=V_1\otimes V_2$. Let us denote $f^*\mathcal{O}_{\mathbb{P}_1}(a)\otimes g^*\mathcal{O}_{\mathbb{P}_1}(b)$ by $\mathcal{O}_Q(a,b)$ and $E\otimes\mathcal{O}_Q(a,b)$ by $E(a,b)$ for coherent sheaves E on Q, where f and g are the projections from Q to each factor. Then the canonical line bundle K_Q of Q is $\mathcal{O}_Q(-2,-2)$.

Definition 2.1. For a fixed ample line bundle H on Q, a torsion-free sheaf E of rank r on Q is called *stable* (respectively, *semi-stable*) with respect to H if

$$\frac{\chi(F\otimes\mathcal{O}_Q(mH))}{r'}<(\textit{respectively},\leq)\frac{\chi(E\otimes\mathcal{O}_Q(mH))}{r},$$

for m big enough and for all non-zero subsheaves $F\subset E$ of rank r'.

Let $\overline{M}(k)$ be the moduli space of semi-stable sheaves of rank 2 with Chern classes $c_1=\mathcal{O}_Q(1,1)$ and $c_2=k$ on Q with respect to the ample line bundle $H=\mathcal{O}_Q(1,1)$. The existence and the projectivity of $\overline{M}(k)$ were shown in [3]. By the Bogomolov theorem [7, Theorem 12.1.1], we also know that $\overline{M}(k)=\emptyset$ for $k\leq 0$. Note that $E\simeq E^*(1,1)$, and by the Riemann–Roch theorem,

$$\chi_E(m):=\chi(E\otimes\mathcal{O}_Q(mH))=2m^2+6m+5-k,$$

for $E\in\overline{M}(k)$. For the case when $k=1$, the following statement can be easily derived.

Lemma 2.2. *$\overline{M}(1)$ is the one-point space whose point is the strictly semi-stable vector bundle*

$$E_0=\mathcal{O}_Q(1,0)\oplus\mathcal{O}_Q(0,1).$$

Proof. Let $E\in\overline{M}(1)$ be a stable bundle. Since $\chi(E)=4$ and $h^2(E)=h^0(E^*(-2,-2))=h^0(E(-3,-3))=0$ due to the stability of E, we have $h^0(E)\geq 4$. From a section s in $H^0(E)$, we have the following exact sequence:

$$0\to\mathcal{O}_Q\xrightarrow{s}E\to I_Z(1,1)\to 0,$$

where Z is a locally complete intersection in Q of codimension 2, i.e., Z is a 0-cycle of Q and I_Z is its ideal sheaf. The length of Z is $c_2(E)=1$, and so Z is a point. If we tensor the sequence with $\mathcal{O}_Q(-1,0)$, it can be shown that $h^0(E(-1,0))=1$, which is a contradiction to the stability of E. If E is strictly semi-stable, it should be fitted into the following sequence:

$$0\to\mathcal{O}_Q(a,1-a)\to E\to\mathcal{O}_Q(1-a,a)\to 0,$$

where $a=0$ or 1. Since $\mathrm{Ext}^1(\mathcal{O}_Q(a,1-a),\mathcal{O}_Q(1-a,a))$ is trivial, E must be the direct sum of $\mathcal{O}_Q(1,0)$ and $\mathcal{O}_Q(0,1)$. □

Now let us deal with the case $k=2$. Note that the stability and semi-stability conditions are equivalent since $\chi_E(m)/2=m^2+3m+3/2$ is not an integer. In particular, $\overline{M}(2)$ is a projective space whose points correspond to isomorphism classes of stable sheaves on Q with given numerical invariants. Let E be a sheaf in $\overline{M}(2)$. For a subsheaf $\mathcal{O}_Q(a,b)$ of E, we have $a+b<1$ by the stability condition. Similarly as above, we can obtain that $h^0(E)\geq 3$. Hence, E admits the following extension:

$$0\to\mathcal{O}_Q\to E\to I_Z(1,1)\to 0, \tag{1}$$

where Z is a zero-dimensional subscheme of Q with length $c_2(E)=2$. In particular, we have $h^0(E)=3$ and $h^1(E)=0$.

Remark 2.3. Let l be the line in $\mathbb{P}_3$ containing Z, and assume that l is contained in Q with the ideal sheaf $\mathcal{O}_Q(-1,0)$. If we tensor the sequence (1) with $\mathcal{O}_Q(0,-1)$ and take the long exact sequence of cohomology, we have $h^0(E(0,-1))=h^0(I_Z(1,0))=1$, contradicting the stability of E. So, if E is stable, then the line l containing Z intersects with Q only at Z.

By a standard computation, we have $\mathrm{Ext}^2(E,E)=0$ for any $E\in\overline{M}(2)$, applying $\mathrm{Hom}(\cdot,E)$ to the sequence (1). Thus, $\overline{M}(2)$ is smooth and three-dimensional.

Lemma 2.4. *The sheaves in the extension* (1) *are all stable if Z is not contained in any line on Q.*

Proof. From the condition on the numerical invariants of E, the only possibilities for the subbundle $\mathcal{O}_Q(a,b)\subset E$ is $\mathcal{O}_Q$ or $\mathcal{O}_Q(a,1-a)$ with $a=0,1$. The second case is impossible due to the condition on Z. □

Let us assume that Z is reduced, and define $\mathbb{P}(Z):=\mathbb{P}\,\mathrm{Ext}^1(I_Z(1,1),\mathcal{O}_Q)$ parameterizing the extensions of the form (1) with fixed Z. Since we have

$$\mathrm{Ext}^1(I_Z(1,1),\mathcal{O}_Q)\simeq H^1(I_Z(-1,-1))^*\simeq H^0(\mathcal{O}_Z)^*,$$

$\mathbb{P}(Z)$ is isomorphic to $\mathbb{P}_1$. From the isomorphism

$$\mathcal{E}xt^1(I_Z(1,1),\mathcal{O}_Q)\simeq\mathcal{O}_{z_1}\oplus\mathcal{O}_{z_2},$$

where $Z=\{z_1,z_2\}$, we can denote (c_1,c_2) for the coordinates of $\mathbb{P}(Z)$. As stated in the case of the projective plane [5, Section 10.3], we can obtain the following lemma.

Lemma 2.5. *An extension (c_1, c_2) gives a bundle if and only if all $c_i \neq 0$.*

Proof. See Lemma 5.1.2 in Chapter 1 of [8] or [1] $\square$

For the non-locally free sheaves in $\overline{M}(2)$, we obtain the following observation.

Lemma 2.6. *The set of non-locally free sheaves in $\overline{M}(2)$ is parameterized by $Q \subset \mathbb{P}_3$.*

Proof. Let E be a non-locally free sheaf in $\overline{M}(2)$; then we have the following exact sequence:

$$0 \to E \to E^{**} \to \mathcal{O}_Z \to 0, \tag{2}$$

where the first map is the natural map to the double dual E^{**} and Z is a zero-dimensional subscheme of Q. Since E^{**} is semi-stable and $c_2(E^{**})$ is less than $c_2(E)$, the only possibility for $c_2(E^{**})$ is 1 (recall that $\overline{M}(0) = \emptyset$), and so we have $E^{**} \simeq E_0 = \mathcal{O}_Q(1, 0) \oplus \mathcal{O}_Q(0, 1)$ by the Lemma 2.2. In particular, the length of Z is $c_2(E) - c_2(E_0) = 1$.

Now, conversely, let $f : E_0 \twoheadrightarrow \mathcal{O}_{p_E}$ be a surjection with kernel E, where p_E is a point in Q. We can denote f by (f_1, f_2), where $f_i : \mathcal{O}_Q(i-1, 2-i) \to \mathcal{O}_{p_E}$, i.e., the parameterizing space of such surjections is $\mathbb{P}_1$ with the coordinates (f_1, f_2). If $f_2 = 0$, then $\ker(f)$ is decomposed to $\mathcal{O}_Q(1, 0) \oplus I_{p_E}(0, 1)$, which is not stable. Similarly, we have an unstable kernel when $f_1 = 0$. Let us assume that $f_i \neq 0$ for all i. Then we have the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I_{p_E}(1,0) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \mathcal{O}_Q(0,1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_Q(1,0) \arrow[r, "s"] \arrow[d] & E_0 \arrow[r] \arrow[d] & \mathcal{O}_Q(0,1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{p_E} \arrow[r] \arrow[d] & \mathcal{O}_{p_E} \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0, & &
\end{tikzcd} \tag{3}$$

where E is a non-trivial extension. If $I_Z(a, b)$ is a subsheaf of E, then we have a non-trivial morphism from $I_Z(a, b)$ to either $I_{p_E}(1, 0)$ or $\mathcal{O}_Q(0, 1)$. In either case, we have $a, b, a+b \leq 1$, and so E is stable. Since the dimension of $\mathrm{Ext}^1(\mathcal{O}_Q(0, 1), I_{p_E}(1, 0))$ is 1, the non-trivial extension E is uniquely determined, i.e., we have only one non-locally free sheaf on Q associated to a point $p_E \in Q$. Thus we get the assertion. $\square$

Now, let E be a stable bundle in $\overline{M}(2)$, and let s be a section in $H^0(E)$. From s, we have an exact sequence of type (1), and so can consider a morphism

$$\psi_E : \mathbb{P}H^0(E) \simeq \mathbb{P}_2 \to Gr(1, \mathbb{P}_3) \subset \mathbb{P}_5,$$

sending a section s of E to the line containing Z in $\mathbb{P}_3$, where $Gr(1, \mathbb{P}_3)$ is the Grassmannian variety parameterizing projective lines in $\mathbb{P}_3$.

Remark 2.7. In the previous proof, if we choose a section of E_0 whose zero is $q \neq p_E$, then we have an exact sequence (1), where $Z = \{p_E, q\}$. In particular, for a non-bundle E, the image of ψ_E is the plane in $Gr(1, \mathbb{P}_3)$ whose points correspond to lines passing through the singularity point p_E.

Let E be a stable bundle in $\overline{M}(2)$ with the exact sequence (1). Since $c_1(E) = \wedge^2 E = \mathcal{O}_Q(1, 1)$, we have a determinant map

$$\lambda_E : \wedge^2 H^0(E) \to H^0(\wedge^2 E) = H^0(\mathcal{O}_Q(1, 1)).$$

Note that $\dim \wedge^2 H^0(E) = 3$ and $h^0(\mathcal{O}_Q(1, 1)) = 4$.

Lemma 2.8. *The determinant map λ_E is injective.*

Proof. Let us assume that λ_E is not injective. Since every element in $\wedge^2 H^0(E)$ is decomposable, there exist two sections s_1 and s_2 of E such that $s_1 \wedge s_2$ is a non-trivial element in $\ker(\lambda_E)$. Then s_1 and s_2 generate a subsheaf F of E with $h^0(F) \geq 2$. Hence, F is of the form $I_{Z'}(a, b)$, where Z' is a 0-cycle on Q and $a, b \geq 0$. From the stability condition of E, we have $F \simeq \mathcal{O}_Q(a, 1-a)$ with $a = 0$ or 1, which is not possible since Z is not contained in any line on Q (Remark 2.3). $\square$

Let p_E be the point in $\mathbb{P}_3 \simeq \mathbb{P}H^0(\mathcal{O}_Q(1,1))^*$ corresponding to the cokernel of λ. Note that $\mathbb{P}H^0(\mathcal{O}_Q(1,1))^*$ is the original ambient space containing Q. Let us consider the following exact sequence:

$$0 \to H^0(I_Z(1,1)) \to H^0(\mathcal{O}_Q(1,1)) \to H^0(\mathcal{O}_Z) \to 0.$$

Lemma 2.4 together with sequence (1) implies that $H^0(E)$ can be expressed as the direct sum of $H^0(\mathcal{O}_Q)$ and $H^0(I_Z(1,1))$, which would give the following identification:

$$\wedge^2 H^0(E) \simeq [H^0(\mathcal{O}_Q) \otimes H^0(I_Z(1,1))] \oplus [\wedge^2 H^0(I_Z(1,1))]. \tag{4}$$

From this identification, clearly, we have

$$H^0(I_Z(1,1)) \subset \lambda_E(\wedge^2 H^0(E)). \tag{5}$$

In other words, the dual of the cokernel of λ is contained in $H^0(\mathcal{O}_Z)^*$. Note that $\mathbb{P}H^0(\mathcal{O}_Z)^*$ is a line in $\mathbb{P}_3$ passing through Z. Thus the line passing through Z also contains p_E. With the previous remark, we get the following lemma.

Lemma 2.9. *ψ_E is a linear embedding of $\mathbb{P}H^0(E)$ into $Gr(1,\mathbb{P}_3) \subset \mathbb{P}_5$. Moreover, the image corresponds to the set of lines passing through one point p_E in $\mathbb{P}_3$.*

Now, let us define a map

$$\Psi : \overline{M}(2) \to \mathbb{P}_3,$$

sending E to p_E, where p_E is the unique point in $\mathbb{P}_3$, passed by the lines in the image of ψ_E.

Proposition 2.10. *$\Psi : \overline{M}(2) \to \mathbb{P}_3$ is an isomorphism.*

Proof. Let p be a point in $\mathbb{P}_3$. If Z is a 0-cycle of length 2 on Q such that the line l passing through Z contains p and is not contained in Q, then we can identify l with $\mathbb{P}(Z)$. If $E \in \mathbb{P}(Z)$ is the sheaf corresponding to the point p, then we have $p = p_E$. Thus Ψ is surjective.

Moreover, assume that p is not in Q. If we take the projection $\pi_p : \mathbb{P}_3 \dashrightarrow \mathbb{P}_2$ from p, then the restriction map $\pi_p : Q \to \mathbb{P}_2$ is a finite morphism of degree 2. Again, if we take the pull-back of $\Omega_{\mathbb{P}_2}(2)$ on Q, we get a stable vector bundle E on Q with the Chern classes $c_1 = \mathcal{O}_Q(1,1)$ and $c_2 = 2$, contained in $\mathbb{P}(Z)$, where Z is a 0-cycle of length 2 on Q for which the line containing Z passes through p (this will be proved later in the Lemma 4.3). In particular, we have $p_E = p$, and so this defines an inverse map of Ψ. Hence, Ψ is a birational morphism, and it is an isomorphism over the stable vector bundles.

Since Ψ is also an isomorphism over the non-bundles from the Lemma 2.6, Ψ is an isomorphism. $\square$

Remark 2.11. From the identification of $\overline{M}(2)$ with $\mathbb{P}_3$, we know that $\mathbb{P}(Z)$ is exactly the secant line of Q in $\mathbb{P}_3$ passing through Z.

Remark 2.12. Alexander Kuznetsov pointed out that the moduli space $\overline{M}(2;1,1,1)$ of stable sheaves on $\mathbb{P}_3$ of rank 2 with the Chern classes $(c_1,c_2,c_3) = (1,1,1)$ is isomorphic to $\mathbb{P}_3$, whose points correspond to the cokernels E of $\mathcal{O}_{\mathbb{P}_3} \to T_{\mathbb{P}_3}(-1)$, where $T_{\mathbb{P}_3}$ is the tangent bundle of $\mathbb{P}_3$. Note that $h^0(\mathbb{P}_3, T_{\mathbb{P}_3}(-1)) = 4$. The restriction map from $\overline{M}(2;1,1,1) \to \overline{M}(2)$ turns out to be an isomorphism.

3. Jumping conics

Definition 3.1. Let E be a stable sheaf in $\overline{M}(2)$, and let H be a hyperplane in $\mathbb{P}_3$. A conic $C_H = Q \cap H$ on Q is called *a jumping conic* of E if the splitting type of $E|_{C_H}$ is different from the generic splitting type.

Proposition 3.2. *For a stable bundle $E \in \overline{M}(2)$, the set of jumping conics forms a hyperplane $H_E \subset \mathbb{P}_3^*$ corresponding to $p_E \in \mathbb{P}_3$. In particular, Ψ can be also defined by sending E to the set of jumping conics of E.*

Proof. Assume that E is locally free. If H is a hyperplane, which does not contain p_E, then we can choose a line l through p_E, with the unique intersection point with C_H, say q. Let $Z = l \cap Q$, and then E is fitted into the exact sequence (1), i.e., $E \in l = \mathbb{P}(Z)$. If we tensor the sequence with $\mathcal{O}_{C_H}$, then $E|_{C_H}$ lies in $\mathrm{Ext}^1(\mathcal{O}_{C_H}(h-q), \mathcal{O}_{C_H}(q)) = 0$, where h is a hyperplane section of $C_H \subset \mathbb{P}_3$, i.e.,

$$E|_{C_H} \simeq \mathcal{O}_{C_H}(q) \oplus \mathcal{O}_{C_H}(h-q). \tag{6}$$

If H contains p_E, let us choose a line $l \subset H$ containing p_E. If we let $Z = l \cap Q$ again, we have $E \in l = \mathbb{P}(Z)$. But, in this case, $E|_{C_H}$ lies in $\mathrm{Ext}^1(\mathcal{O}_{C_H}, \mathcal{O}_{C_H}(h)) = 0$, i.e.,

$$E|_{C_H} \simeq \mathcal{O}_{C_H}(h) \oplus \mathcal{O}_{C_H}. \tag{7}$$

Hence, jumping conics of E can be characterized by $h^0(E(-1)|_{C_H}) \neq 0$. $\square$

Similarly, when E is not locally free, we can derive the same cohomological criterion for the jumping conics in the proof of the above proposition.

4. Restriction map

Let C be a non-hyperelliptic curve of genus 4. It is embedded into $\mathbb{P}H^0(K_C)^* \simeq \mathbb{P}_3$, and there is a unique quadric surface $Q \subset \mathbb{P}_3$ containing C. Let g_3^1 and h_3^1 be the two trigonal line bundles in $\Theta \subset \mathrm{Pic}^3(C)$ such that $g_3^1 \otimes h_3^1 = \mathcal{O}_C(K_C)$. Note that Q is smooth if and only if $|g_3^1| \neq |h_3^1|$. From now on, we assume that Q is smooth. Let $SU_C(2, K_C)$ be the moduli space of semi-stable vector bundles of rank 2 with canonical determinant over C, and let $\mathcal{W}^r$ be the closure of the following set:

$$\{E \in SU_C(2, K_C) \mid h^0(C, E) \geq r + 1\}. \tag{8}$$

Then we have the following inclusions [9, Section 3] on the Brill–Noether loci:

$$SU_C(2, K_C) \supset \mathcal{W} \supset \mathcal{W}^1 \supset \mathcal{W}^2 \supset \mathcal{W}^3 \supset \mathcal{W}^4 = \emptyset, \tag{9}$$

where $\mathcal{W} = \mathcal{W}^0$. Many geometric descriptions of these Brill–Noether loci have been investigated in [9].

Since C is a divisor of Q with the divisor class $(3, 3)$, we have the exact sequence

$$0 \to \mathcal{O}_Q(-3, -3) \to \mathcal{O}_Q \to \mathcal{O}_C \to 0. \tag{10}$$

Twisting the sequence (10) with a stable vector bundle $E \in \overline{M}(2)$, we obtain that $h^0(C, E|_C) = h^0(Q, E) = 3$, since $h^1(Q, E(-3, -3)) = h^1(Q, E^*(1, 1)) = h^1(Q, E) = 0$ due to the Serre duality and the fact that $E \simeq E^*(1, 1)$.

Lemma 4.1. *For a stable vector bundle $E \in \overline{M}(2)$, its restriction to C, $E|_C$, is stable, and so we have a rational map*

$$\Phi : \overline{M}(2) \dashrightarrow \mathcal{W}^2$$

sending E to $E|_C$.

Proof. Since the embedding of C into $\mathbb{P}_3$ is canonical, the restriction of $\wedge^2 E \simeq \mathcal{O}_Q(1, 1)$ is $\mathcal{O}_C(K_C)$, i.e., $\det(E|_C) = \mathcal{O}_C(K_C)$. If we tensor the exact sequence (1) by $\mathcal{O}_C$, we have

$$0 \to \mathcal{O}_C(D) \to E|_C \to \mathcal{O}_C(K_C - D) \to 0,$$

where $D = Z \cap C$ as a scheme with $l(D) \leq l(Z) = 2$. Suppose that there exists a subbundle $\mathcal{O}_C(D') \subset E|_C$ with $\deg(D') = d' \geq 3$. If the natural composite $f : \mathcal{O}_C(D') \to \mathcal{O}_C(K_C - D)$ is zero, then $\mathcal{O}_C(D') \subset \mathcal{O}_C(D)$, which is not possible. Thus f is not zero and must be injective, so d' can be at most 6. Since $h^0(E|_C) = 3$ and $h^0(\mathcal{O}_C(K_C - D')) \leq 2$, we get that $H^0(\mathcal{O}_C(D'))$ is not trivial. Now, we can assume that D' is effective and is contained in the zeros of a section in $H^0(E|_C)$. Note that $H^0(E) \simeq H^0(E|_C)$, and so every section of $E|_C$ comes as the restriction of a section of E. Since the zero of a section of E has only two points as its support, the degree of D' must be less than 3. Hence, $E|_C$ is stable. □

Lemma 4.2 (*[9, Lemma 6.6]*)**.** *For a general E in $\mathcal{W}^2$, the determinant map*

$$\lambda_E : \wedge^2 H^0(E) \to H^0(K_C)$$

is injective.

Proof. As in the Lemma 2.8, let us assume that s_1 and s_2 are two sections of E for which $s_1 \wedge s_2$ is a non-trivial element in $\ker(\lambda_E)$. They would generate a subbundle $L \subset E$ with $h^0(L) \geq 2$ and $\deg(L) \leq 2$, contrary to the fact that C is non-hyperelliptic. □

Denote by p_E the point in $\mathbb{P}H^0(K_C)^* \simeq \mathbb{P}_3$ corresponding to the cokernel of λ_E. Sending E to p_E would define a map

$$\tau : \mathcal{W}^2 \dashrightarrow \mathbb{P}_3.$$

Moreover, we have the following diagram:

$$\begin{tikzcd} C \arrow[r] & \mathbb{P}H^0(K_C)^* \simeq \mathbb{P}_3 \arrow[d, "\pi_E"] \\ & \mathbb{P}(\wedge^2 H^0(E)^*) \simeq \mathbb{P}_2, \end{tikzcd}$$

where π_E is the projection from p_E. From the identification of $\mathbb{P}H^0(K_C)^* \simeq \mathbb{P}H^0(\mathcal{O}_Q(1, 1))^*$ and $\mathbb{P}(\wedge^2 H^0(E|_C)^*) \simeq \mathbb{P}(\wedge^2 H^0(E)^*)$ for E a stable bundle in $\overline{M}(2)$, we know that $p_{E|_C}$ does not lie on Q. Hence, p_E does not lie on Q for general $E \in \mathcal{W}^2$.

Let E be a general vector bundle in $\mathcal{W}^2$. Note that $\mathbb{P}(\wedge^2 H^0(E)^*)$ is the Grassmannian $Gr(H^0(E), 2)$, parameterizing two-dimensional quotient vector spaces of $H^0(E)$, and its universal quotient bundle is $\Omega_{\mathbb{P}_2}(2)$. From the construction, we have $\pi_E^*(\Omega_{\mathbb{P}_2}(2))|_C \simeq E$, and the Chern classes of $\pi_E^*(\Omega_{\mathbb{P}_2}(2))|_Q$ are $(c_1, c_2) = (\mathcal{O}_Q(1, 1), 2)$, since $\pi_E : Q \to \mathbb{P}_2$ is a finite map of degree 2.

Lemma 4.3. $E_Q := \pi_E^*(\Omega_{\mathbb{P}_2}(2))|_Q$ *is a stable vector bundle on Q.*

Proof. If we take the pull-back $(\pi_E|_Q)^*$ of the following sequence,

$$0 \to \mathcal{O}_{\mathbb{P}_2} \to \Omega_{\mathbb{P}_2}(2) \to I_p(1) \to 0, \tag{11}$$

where I_p is the ideal sheaf of a point $p \in \mathbb{P}_2$, then E_Q is a non-trivial extension class in $\mathrm{Ext}^1(I_Z(1,1), \mathcal{O}_Q)$, where $Z = \pi_E^{-1}(p)$. Any non-trivial element in this extension is stable due to the Lemma 2.4. □

Corollary 4.4. *The restriction map Φ is birational.*

Remark 4.5. Note that g_3^1 and h_3^1 are isomorphic to $\mathcal{O}_Q(1,0)|_C$ and $\mathcal{O}_Q(0,1)|_C$. In particular, $E_0|_C = g_3^1 \oplus h_3^1$ and $h^0(E_0) = h^0(E_0|_C) = 4$. As pointed out in [9, Remark 6.5], $g_3^1 \oplus h_3^1$ can be considered as the unique point of $\mathcal{W}^3$.

Remark 4.6. Let E be a non-bundle in $\overline{M}(2)$, and let p be its singularity point. If $p \notin C$, then $E|_C \simeq E_0|_C \in \mathcal{W}^3$. If $p \in C$, then $E|_C$ is not torsion free. In particular, the indeterminacy locus of Φ is exactly $C \subset \mathbb{P}_3 \simeq \overline{M}(2)$.

Proposition 4.7. *The map Φ is given by the complete linear system $|I_C(3)|$.*

Proof. We know that Φ is an isomorphism on $\mathbb{P}_3 \backslash Q$ and that it sends $Q \backslash C$ to one point $E_0|_C$. Let H be a general hyperplane in $\mathbb{P}_3$. The intersection of H with C is six points on Q, say $P_1, \ldots, P_6$, and these points lie on a conic $C_2 = H \cap Q$. The restriction of Φ to H is not defined on the P_i's, and it maps the other points of C_2 to $E_0|_C$. Note that any general straight line on H maps to a nodal curve in $\mathcal{W}^2$ birationally. Let l be a line on H passing through P_1, but not the other P_i's. The restriction of Φ to l is a map from $\mathbb{P}_1$ to a projective space, and thus it is well defined everywhere, in particular, at P_1. In other words, when we blow up H at P_1, then Φ can be extended to the exceptional divisor of the blow-up. Let S_6 be the blow-up of H at the P_i's, and then we obtain a morphism f_H from S_6 to $\mathcal{W}^2$.

$$\begin{tikzcd} & S_6 \arrow[dl] \arrow[dr, "f_H"] & \\ H \arrow[rr, dashed] & & \mathcal{W}^2 \end{tikzcd} \tag{12}$$

The proper transform of C_2 in S_6 is a line l_H, and f_H contracts l_H to a point. Since $f_H(S_6)$ is a cubic surface in $\mathbb{P}_3$, the degree of the restriction of Φ to H is 3, and so is the degree of Φ.

Recall that the indeterminacy of Φ is C, and note that $h^0(I_C(3)) = 5$. Since Φ is a birational map, the dimension of the linear system $V \subset H^0(I_C(3))$ defining Φ is at least $4 = \dim \overline{M}(2) + 1$. Assume that $\dim V = 4$. Then Φ is a birational map from $\mathbb{P}_3 = \overline{M}(2)$ to $\mathbb{P}_3$ sending $Q \backslash C$ to one point. But this is impossible. Hence Φ must be given by the complete linear system $|I_C(3)|$. □

Remark 4.8. The image of $\mathbb{P}_3$ via $|I_C(3)|$ is known to be the *Donagi–Izadi cubic threefold* in $\mathbb{P}_4^*$ [2,6,9]. This is singular, with a nodal point $P = E_0|_C$.

Let π_P be the projection from $\mathbb{P}_4^*$ at P and then we have the following commutative diagram:

$$\begin{tikzcd} \overline{M}(2) \arrow[r, dashed, "\Phi"] \arrow[d, "\Psi"', "\wr"] & \mathcal{W}^2 \arrow[d, dashed, "\pi_P"] \arrow[dl, dashed, "\tau"'] \\ \mathbb{P}_3 \arrow[r, "f_Q"', "\sim"] & \mathbb{P}_3^*, \end{tikzcd} \tag{13}$$

where the map f_Q is defined as follows: let H' be a hyperplane in $\mathbb{P}_3^*$. Then H' pulls back via π_P to a hyperplane in $\mathbb{P}_4^*$ containing P, and then to Q and a residual hyperplane $H \subset \mathbb{P}_3$ by Φ.

Let p_E be a point in $\mathbb{P}_3$ corresponding to $E \in \overline{M}(2)$, and let H_E' be its hyperplane in $\mathbb{P}_3^*$. As above, we can assign a residual hyperplane $H_E \subset \mathbb{P}_3$ and a conic $C_E = H_E \cap Q$ to E. Simply, f_Q is a polar map given by

$$[x_0, \ldots, x_3] \mapsto \left[\frac{\partial f}{\partial t_0}(x), \ldots, \frac{\partial f}{\partial t_3}(x) \right], \tag{14}$$

where f is the homogeneous polynomial of degree 2 defining Q. The hyperplane $H_E \subset \mathbb{P}_3$, corresponding to $f_Q(p_E)$, is given by the equation

$$\sum_{i=0}^{3} \frac{\partial f}{\partial t_i}(p_E) t_i = 0, \tag{15}$$

and, from the Euler formula, it is clear that $p_E \in Q$ is equivalent to $p_E \in H_E$. Assume that $p_E \notin Q$. Let us recall that $C_E = H_E \cap Q$ is the set of points $p \in Q$ for which p_E is contained in the tangent plane of Q at p. In particular, E fits into an extension (1), where Z is a point p with multiplicity 2. In other words, there exists a section s of E whose zero is p with multiplicity 2. We can have the same argument for the case when $p_E \in Q$.

Proposition 4.9. *Let E be a stable sheaf in $\overline{M}(2)$. The set of points that are the zero with multiplicity 2 for some section of E forms a conic C_E in Q. This defines an isomorphism*

$$f_Q \circ \Psi : \overline{M}(2) \to \mathbb{P}_3^*.$$

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